UDC 536.46:533.6+534.222.2

UNSTEADY PROPAGATION OF THE COMBUSTION WAVE IN A MEDIUM WITH SLOWLY VARYING PARAMETERS

PMM Vol.42, № 3, 1978, pp. 450-457

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(Received January 21, 1977)

Propagation of an unsteady combustion wave in an inhomogeneous unsteady gaseous medium with slowly varying parameters is investigated. Distribution of reagent concentration and of temperature in the system is assumed to be similar. Equations defining the evolution of a three-dimensional combustion wave are derived, and a number of examples is considered.

The classical problem of the theory of combustion is that of steady propagation of a combustion wave in a homogeneous medium [1]. However in a number of cases the distribution of thermophysical properties of the medium depends not only on coordinates but, also, on time. The propagation of a combustion wave is then unsteady and may be accompanied by flame front distortions.

To study the effect of time and space inhomogeneities on the combustion wave we shall consider the propagation of a steady reaction wave in a medium whose thermophysical parameters are variable. For the analytical investigation of this problem we assume that these inhomogeneities vary substantially in space and time intervals $O(\varepsilon^{-1})$ that are considerable in comparison with the scales of the heat layer characteristic thickness $\varkappa / u_{st}$ and its characteristic rearrangement time $(\varkappa / u_{st}^2)$, respectively. In these expressions $\varkappa$ is the thermal dissipativity of gas, u_{st} is the steady propagation rate of the combustion wave, and $0 < \varepsilon \ll 1$ is a small dimensionless parameter that defines the variation rate of thermophysical parameters. We assume that variation of the medium properties can be of the order of (0.1).

Interaction between an acoustic wave and a combustion wave can be cited as an example, taking into account that in real systems the acoustic wave length Λ may considerably exceed the thickness of the steady heat layer, so that basic variations of temperature and concentration take place in a pressure field that is uniform in space and depends on time. The following relationships must then be satisfied:

$$c/\omega \sim \Lambda \gg \varkappa/u_{st} \qquad (0.1)$$

where c is the speed of sound and ω is the acoustic wave frequency.

The results of the present investigation can be applied for defining the interaction between a combustion wave and a sonic wave of fairly low fequency for which the following relationship is valid

$$\omega \ll u_{st}^2/\varkappa \qquad (0.2)$$

Note that for the characteristic values $u_{st} = 10^2$ cm, $\varkappa = 10^{-1}$ cm²/sec, $c = 10^5$ cm/sec and $\varkappa/u_{st} = 10^{-3}$ cm inequality (0.1) is valid when $\omega \ll 10^8$ Hz, while inequality (0.2) holds when $\omega \ll 10^5$ Hz. Thus the investigation covers a fairly wide range of frequencies.

A model problem of combustion wave propagation in gas for which the Lewis number is unity is considered below. It is shown that in many cases distortion of the wave profile may be apparent already in the zero approximation with respect to ε. Variation of the temperature field in time and space, as well as the velocity of motion of isothermal surfaces are determined.

1. Basic equations. The model problem of unsteady wave propagation in a medium may be defined on a number of assumptions by the following demensionless equation:

$$\frac{\partial \Theta}{\partial t} = \frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Y^2} + \frac{\partial^2 \Theta}{\partial Z^2} + f^2(\tau, x, y, z) F(\Theta) \tag{1.1}$$

where Θ is the temperature; X, Y, and Z are space coordinates; $x = X\varepsilon$, $y = Y\varepsilon$, $z = Z\varepsilon$, $\tau = \varepsilon t$, and $0 < \varepsilon \ll 1$; function f defines temporal and spatial inhomogeneities ($f \equiv 1$ when $t = 0$); $F(\Theta)$ is a nonlinear function that defines the dependence of the heat release rate on temperature. In what follows we assume that $f = O(1)$.

The quantities $\varkappa / u_{st}^2$ and $\varkappa / u_{st}$ are taken as the scales of time and space variables, respectively.

Usually

$$0 \leqslant \Theta \leqslant 1, \quad F(0) = F(1) = 0 \tag{1.2}$$

It is assumed that when $t < 0$ a stable steady combustion wave propagates in the gas along the axis X ($f \equiv 1$).

Then for $t < 0$ we have

$$\frac{d^2\varphi}{ds^2} + \frac{d\varphi}{ds} + F(\varphi) = 0 \tag{1.3}$$

$$\varphi(-\infty) = 1, \quad \varphi(+\infty) = 0 \tag{1.4}$$

$$\theta(X, t) = \varphi(s), \quad s = X - t$$

It was shown in [2, 3] that the problem (1.3), (1.4) has a solution when

$$F'(0) > 0, \quad F'(1) > 0; \quad \Theta \in [0, a), \quad F(\Theta) \geqslant 0$$
$$\Theta \in (0, 1], \quad F(\Theta) \leqslant 0; \quad 0 < a \leqslant 1 \tag{1.5}$$

or under conditions [4]

$$F'(0) < 0; \quad \Theta \in [0, a), \quad F(\Theta) \leqslant 0$$
$$\Theta \in [a, 1), \quad F(\Theta) \geqslant 0; \quad 0 \leqslant a \leqslant 1; \quad \int_0^1 F(\Theta)\, d\Theta > 0 \tag{1.6}$$

and also when [5]

$$\Theta \in [0, \delta], \quad F(\Theta) \equiv 0; \quad \Theta \in [\delta, 1)$$
$$F(\Theta) > 0; \quad 0 < \delta < 1 \tag{1.7}$$

For the analysis it is important to know the asymptotics of $\varphi(s)$ when $s \to +\infty$. It follows from (1.5) – (1.7) and (1.3) that $\varphi(s \to +\infty) = O(\exp(-\alpha s))$, $\alpha \geqslant 1$. Hence the integrals (1.8)

$$I_1 = \int_{-\infty}^{+\infty} \left(\frac{d\varphi}{dx}\right)^2 e^x dx; \quad I_2 = \int_{-\infty}^{+\infty} \left(\frac{d\varphi}{dx}\right)^2 xe^x dx \tag{1.8}$$

which appear subsequently, do exist.

In a number of cases integrals I_1 and I_2 can be approximately estimated. Thus often in the theory of combustion function $F(\varphi(s)) \sim O(1)$ only in the interval considerably smaller than unity, while for the other values $\varphi - |F(\varphi)| \ll 1$. Hence the approximation [6]

$$\varphi(s) = \begin{cases} 1, & s < 0 \\ e^{-s} & s > 0 \end{cases}$$

seems fairly reasonable. The quantity $I = I_1^{-1} I_2$ that appears subsequently is approximately

$$I = 1 + o(1)$$

Usually the solution of (1.3), (1.4) in closed form is not known. However approximate methods can be used in many cases for the determination of $\varphi(s)$. Thus for function $F(\Theta) \sim (1 - \Theta)^n \exp(\beta(\Theta - 1))$ $(\beta \gg 1)$ typical in the theory of combustion, it is possible to use methods of matching asymptotic expansions [7, 8]. Note that the presence of two small parameters $\beta^{-1} \ll 1$ and $\varepsilon \ll 1$ does not lead to an inconsistency of respective expansions, since expansions in β^{-1} are formed in region $X \sim O(\beta^{-1})$, while those in ε are obtained in $X \sim O(\varepsilon^{-1})$.

For Eq. (1.1) we have the following boundary conditions:

$$X = -\infty, \quad \Theta(-\infty, Y, Z, t) = 1 \tag{1.9}$$

$$X = +\infty, \quad \Theta(+\infty, Y, Z, t) = 0.$$

For a finite X the solution of (1.1) must be bounded for any Y, Z, and t. The initial condition for (1.1) is of the form

$$\Theta(X, Y, Z, 0) = \varphi(X) \tag{1.10}$$

2. The method of solution. We seek a solution of (1.1), (1.9), (1.10) of the form

$$\Theta(X, Y, Z, t; \varepsilon) = \Theta_0(\eta, x, y, z, \tau) + \varepsilon\Theta_1(\eta, x, y, z, \tau) + \ldots \tag{2.1}$$

assuming that expansion (2.1) is uniformly valid, i.e. that $\Theta_1 / \Theta_0 \approx O(1)$ for all (x, y, z, τ), where

$$\eta = \xi(x, y, z, \tau; \varepsilon)\varepsilon^{-1}; \quad \xi = \xi_0(x, y, z, \tau) + \varepsilon\xi_1(x, y, z, \tau) + \ldots \tag{2.2}$$

A similar method was used in [9] for obtaining solutions of nonlinear equations of the hyperbolic type and for determining the shape of solitary waves (soletones) over floor of varying shape [10]. The described method was also used in a number of other investigations [11, 12]. Passing from variables (X, Y, Z, t) to (η, x, y, z, τ) by formulas

$$\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2} = |\nabla\xi|^2 \frac{\partial^2}{\partial \eta^2} + \varepsilon\left(2(\Delta\xi\nabla)\frac{\partial}{\partial\eta} + \nabla^2\xi\frac{\partial}{\partial\eta}\right) + \varepsilon^2\nabla^2 \tag{2.3}$$

$$\frac{\partial}{\partial t} = \varepsilon\frac{\partial}{\partial\tau} + \frac{\partial\xi}{\partial\tau}\frac{\partial}{\partial\eta} \tag{2.4}$$

where ∇ and ∇^2 are operators in $\mathbf{r}$ $(\mathbf{r} = (x, y, z))$.

Substituting (2.1), (2.3), and (2.4) into (1.1) and equating terms with like powers of ε, for the first two terms of expansion (2.1) we have

$$|\nabla\xi_0|^2 \frac{\partial^2\Theta_0}{\partial\eta^2} - \frac{\partial\xi_0}{\partial\tau}\frac{\partial\Theta_0}{\partial\eta} + f^2(\mathbf{r},\tau)F(\Theta_0) = 0 \tag{2.5}$$

$$|\nabla\xi_0|^2 \frac{\partial^2\Theta_1}{\partial\eta^2} - \frac{\partial\xi_0}{\partial\tau}\frac{\partial\Theta_1}{\partial\eta} + f^2(\mathbf{r},\tau)\frac{dF(\Theta_0)}{d\Theta}\Theta_1 = \frac{\partial\Theta_0}{\partial\tau} - 2(\nabla\xi_0\nabla)\frac{\partial\Theta_0}{\partial\eta} - \nabla^2\xi_0\frac{\partial\Theta_0}{\partial\eta} - 2(\nabla\xi_0\nabla\xi_1)\frac{\partial^2\Theta_0}{\partial\eta^2} - \frac{\partial\xi_1}{\partial\tau}\frac{\partial\Theta_0}{\partial\eta} \tag{2.6}$$

We assume that solutions of Eqs.(2.5) and (2.6) satisfy boundary conditions (1.9), i.e.

$$\Theta_0(-\infty, \mathbf{r}, \tau) = 1, \quad \Theta_0(+\infty, \mathbf{r}, \tau) = 0 \tag{2.7}$$

$$\Theta_1(-\infty, \mathbf{r}, \tau) = 0, \quad \Theta_1(+\infty, \mathbf{r}, \tau) = 0 \tag{2.8}$$

Equation (2.5) formally conforms to the equation that defines the steady combustion wave (1.3),(1.4). The solution of Eq. (2.5) is of the form

$$\Theta_0(\eta, \mathbf{r}, \tau) = \Psi(f|\nabla\xi_0|^{-1}\eta + a) \tag{2.9}$$

where $a = a(\mathbf{r}, \tau)$ is an arbitrary function and function $\Psi(s)$ satisfies the equation

$$\frac{d^2\Psi}{ds^2} + V\frac{d\Psi}{ds} + F(\Psi) = 0, \quad V = -\frac{\partial\xi_0}{\partial\tau}f^{-1}|\nabla\xi_0|^{-1}$$

$$\Psi(-\infty) = 1, \ \Psi(+\infty) = 0$$

The equation for $\xi_0(\mathbf{r}, \tau)$ of the form $V = 1$ with $\Psi(s) \equiv \varphi(s)$ follows from (1.3) and (1.4). From this we have

$$\frac{\partial\xi_0}{\partial\tau} = -f|\nabla\xi_0|, \quad \xi_0(\mathbf{r}, 0) = x \tag{2.10}$$

For determining function $a(\mathbf{r}, \tau)$ we shall analyze Eq. (2.6). First of all we note that owing to the dependence of function Ψ on the combination $f|\nabla\xi_0|^{-1}\eta + a(\mathbf{r}, \tau)$ in (2.9), function $\xi_1(\mathbf{r}, \tau)$ can be assumed, without loss of generality, equal zero. In what follows we assume $\xi_1 \equiv 0$.

Equation (2.6) is linear with respect to Θ_1 and is a variational equation relative to Eq. (2.5). From the Poincaré 's theorem [13, 14] follows that the fundamental solutions of Eq.(2.6) is represented by functions

$$y_1(\eta) = \frac{\partial}{\partial A}\Phi(\eta + A, B), \quad y_2(\eta) = \frac{\partial}{\partial B}\Phi(\eta + A, B)$$

where $\Phi(\eta + A, B)$ is the general solution of (2.5), and A and B are arbitrary functions of $\mathbf{r}$ and τ. It follows from this and Eq. (2.9) that the solution of the homogeneous equation (2.6) with conditions (2.8) is proportional to $\partial\Theta_0/\partial\eta$. Since the solution $y_2(\eta)$ does not satisfy boundary conditions, i.e. it does not vanish at both ends of the interval $\eta \to -\infty$ to $\eta \to +\infty$, hence for solving the inhomogegeous boundary value problem the right-hand side of (2.6) must be orthogonal with weight $\exp(-|\nabla\xi_0|^{-2}\eta\partial\xi_0/\partial\tau)$ to function $\partial\Theta_0/\partial\eta$ which is the solution of the homogeneous boundary value problem [15], i.e. to the solution of the homogeneous equation conjugate of (2.6). Then

$$\int_{-\infty}^{+\infty} \frac{\partial\Theta_0}{\partial\eta} e^{g\eta}\left(\frac{\partial\Theta_0}{\partial\tau} - 2(\nabla\xi_0\nabla)\frac{\partial\Theta_0}{\partial\eta} - \nabla^2\xi_0\frac{\partial\Theta_0}{\partial\eta}\right)d\eta = 0$$

$$g(\mathbf{r},\tau) = -|\nabla\xi_0|^{-2}\partial\xi_0/\partial\tau$$

Taking into account that

$$\frac{\partial\Theta_0}{\partial\tau} = \left(\frac{\partial g}{\partial\tau}\eta + \frac{\partial a}{\partial\tau}\right)\frac{d\varphi}{ds}$$

$$(\nabla\xi_0\nabla)\frac{\partial\Theta_0}{\partial\eta} = (\nabla\xi_0\nabla g)\frac{d\varphi}{ds} + (\eta g(\nabla\xi_0\nabla g) + g(\nabla\xi_0\nabla a))\frac{d^2\varphi}{ds^2}$$

$$\frac{\partial\Theta_0}{\partial\eta} = g\frac{d\varphi}{ds}$$

we obtain the orthogonality condition of the form

$$\int_{-\infty}^{+\infty} e^{g\eta}\frac{d\varphi}{ds}\left\{\frac{d\varphi}{ds}\left(\eta\frac{\partial g}{\partial\tau} + \frac{\partial a}{\partial\tau}\right) - 2\frac{d\varphi}{ds}(\nabla\xi_0\nabla g) - \right. \tag{2.11}$$

$$\left. 2g\frac{d^2\varphi}{ds^2}(\eta(\nabla\xi_0\nabla)_g + (\nabla\xi_0\nabla a)) - \nabla^2\xi_0 g\frac{d\varphi}{ds}\right\}d\eta = 0$$

Passing in (2.11) from the variable η to s by formula $\eta = (s-a)/g$ and introducing the new unknown function $A(\mathbf{r},\tau)$

$$a(\mathbf{r},\tau) = g(\mathbf{r},\tau)A(\mathbf{r},\tau)$$

we obtain

$$A_\tau\int_{-\infty}^{+\infty}\left(\frac{d\varphi}{ds}\right)^2 e^s ds - 2g\nabla\xi_0\nabla A\int_{-\infty}^{+\infty}\frac{d\varphi}{ds}\frac{d^2\varphi}{ds^2}e^s ds + \tag{2.12}$$

$$g^{-2}\frac{\partial g}{\partial\tau}\int_{-\infty}^{+\infty} e^s\left(\frac{d\varphi}{ds}\right)^2 s\,ds - 2g^{-1}(\nabla\xi_0\nabla)g\int_{-\infty}^{+\infty} e^s\left(\frac{d\varphi}{ds}\right)^2 ds -$$

$$2g^{-1}(\nabla\xi_0\nabla)g\int_{-\infty}^{+\infty} e^s\frac{d\varphi}{ds}\frac{d^2\varphi}{ds^2}s\,ds - \nabla^2\xi_0\int_{-\infty}^{+\infty}\left(\frac{d\varphi}{ds}\right)^2 e^s ds = 0$$

Assuming the existence of integrals appearing in (2.12) and integrating by parts, we obtain

$$\int\limits_{-\infty}^{+\infty} \frac{d\varphi}{ds}\frac{d^2\varphi}{ds^2} e^s\, ds = -\frac{1}{2} I_1, \quad I_1 = \int\limits_{-\infty}^{+\infty} e^s \left(\frac{d\varphi}{ds}\right)^2 ds \tag{2.13}$$

$$\int\limits_{-\infty}^{+\infty} s\frac{d\varphi}{ds}\frac{d^2\varphi}{ds^2} e^s\, ds = \frac{1}{2}\left(\frac{d\varphi}{ds}\right)^2 se^s\Big|_{-\infty}^{+\infty} -$$

$$\frac{1}{2}\int\limits_{-\infty}^{+\infty}\left(\frac{d\varphi}{ds}\right)^2 e^s(1+s)\, ds = -\frac{1}{2}(I_1+I_2)$$

$$I_2 = \int\limits_{-\infty}^{+\infty} e^s s\left(\frac{d\varphi}{ds}\right)^2 ds$$

where I_1 and I_2 are constants.

From (2.12) and (2.13) we have

$$A_\tau + g(\nabla\xi_0\nabla)A + g^{-2}\frac{\partial g}{\partial\tau} I + g^{-1}(\nabla\xi_0\nabla)g(I-1) - \nabla^2\xi_0 = 0, \quad I = I_2 I_1^{-1} \tag{2.14}$$

As an example of application of the obtained results we consider the case of

$$f(\mathbf{r},\ \tau) = \psi(\tau)$$

in which from (2.9), (2.10), and (2.14) we have

$$\xi_0 = x - \int\limits_0^\tau \psi(\tau')\, d\tau', \quad a = I(1-\psi(\tau)) \tag{2.15}$$

$$\Theta_0 = \varphi\left(\psi(\tau)\varepsilon^{-1}\left(x - \int\limits_0^\tau \psi\, d\tau'\right) + I(1-\psi(\tau))\right)$$

3. Discussion of results. The first approximation of the solution of the unsteady problem (1.1), (1.9), and (1.10) was derived in Sect. 2.

The most important characteristic of a combustion wave is the propagation rate of isothermal surfaces. In the steady state ($f \equiv 1$) the velocity of motion of all isothermal surfaces is the same and constant. In the case of nonconstant function, velocity may be determined as implied by (2.9), by formula

$$f(\mathbf{r},\ \tau)\,|\nabla\xi_0|^{-1}\,\xi_0(\mathbf{r},\ \tau) = B \tag{3.1}$$

where B is a constant dependent of the temperature of isothermal surfaces.

It was pointed out in [12] that the solution derived by the method used in this work effectively defines the pattern of temperature distribution only in some neighborhood of the front, while a fair distance from the latter that solution is inadequate. Hence in what follows it is assumed that everywhere $B = O(\varepsilon)$.

Differentiating (3.1) with respect to t we obtain

$$v = \left(\frac{d\mathbf{r}_B}{dt}\right)$$

where $r_B(t)$ is the solution of Eq. (3.1).

Note that it is possible to attempt to obtain a solution of the problem (1.1), (1.9), (1.10) by a different method based on the slow variation of function $f(\mathbf{r}, \tau)$. Assuming that f is fixed we obtain a solution of the problem of the form

$$\Theta = \varphi(f(x - \tau f)\varepsilon^{-1}) \tag{3.2}$$

The inapplicability of this formula can be readily demonstrated. Let us, for instance, consider $f(\mathbf{r}, \tau) = \Phi(\tau)$ and $\Phi(\tau) = O(1)$. The propagation rate of the isotherm $\Phi(\tau)(x - \tau\Phi(\tau)) = B$ is then

$$dx / d\tau = \Phi(\tau) + \tau d\Phi / d\tau + O(\varepsilon) \tag{3.3}$$

When $\tau \to \infty$ and $d\Phi / d\tau = O(1)$ the wave propagation velocity approaches infinity, which is evidently false.

In the considered case from (2.15) and (3.1) we have

$$v = (v_x, 0, 0), \quad v_x = \Phi(\tau) + O(\varepsilon) \tag{3.4}$$

which shows that when $\tau \to \infty$ the velocity remains constant and $d\Phi / d\tau = O(1)$.

Note that in transient modes in $\Phi(+\infty) = \Phi_+ = \text{const}$ both solutions (3.3) and (3.8) when $f(\mathbf{r}, \tau) = \Phi(\tau)$ and $\tau = +\infty$ define the steady solution $\varphi(\Phi_+ (x - \tau\Phi_+)\varepsilon^{-1})$.

In the general case the propagation velocity of the isothermal surface is normal to $f(\mathbf{r}, \tau)$ and does not conform to the direction calculated by the "quasi-steady" formula

$$f(x - \tau f) = \text{const}$$

Note that Eq. (2.10) derived in Sect. 2 can also be used for obtaining approximate solutions of other similar problems such as, for instance, the description of the evolution of a stable combustion wave penetrating a region with variable thermophysical parameters. In that case it is necessary that conditions $\xi_0(\mathbf{r}, \tau) = x$ when $x \to -\infty$.

In the more general case when the transport coefficients depend on $\mathbf{r}$ and τ instead of (1.1) we have

$$\varepsilon a(\mathbf{r}, \tau)\partial\Theta / \partial\tau = \varepsilon^2 \nabla (b(\mathbf{r}, \tau) \nabla \Theta) + c^2(\tau, \mathbf{r}) F(\Theta)$$
$$b > 0, \quad a > 0$$

Then, instead of (2.9) and (2.10) we obtain

$$\Theta_0 = \varphi(cb^{-1/2} |\xi_0|^{-1} \eta + a),$$
$$\partial\xi_0 / \partial\tau = -ca^{-1}b^{1/2} |\nabla\xi_0|$$

The proposed treatment is also suitable in the case of explicit dependence of F on $\mathbf{r}$ and τ.

The author thanks Iu.S. Riazantsev for valuable discussions.

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Translated by J.J.D.